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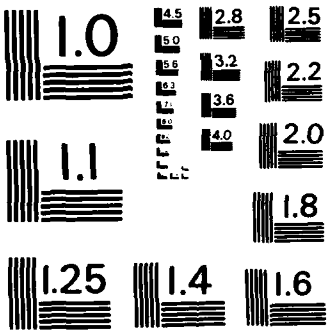

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SOME ASPECTS OF CRITICAL POINT THEORY

Paul H. Rabinowitz

Mathematics Research Center
University of Wisconsin–Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

This paper contains a slightly expanded version of the series of lectures given by the author at DD3, the Third International Symposium on Differential Equations and Differential Geometry held in Changchun, China during parts of August and September of 1982. The lectures describe some of the results obtained using minimax methods in critical point theory during the past several years and give applications of the abstract results to differential equations. In particular existence theorems are obtained for many boundary value problems for semilinear elliptic partial differential equations.

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*Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53705

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SOME ASPECTS OF CRITICAL POINT THEORY

Paul H. Rabinowitz*

The main goal of these lectures is to describe some of the research done on minimax methods in critical point theory during the past several years. A variety of abstract critical point theorems will be stated and proved and applications of these results will be made to differential equations. Due to time limitations we will confine our applications to existence theorems for semilinear elliptic boundary value problems.

To briefly describe the abstract situation that is treated, let E be a real Banach space and $I \in C^1(E,\mathbb{R})$. The Frechet derivative of I at u will be denoted by $I'(u)$. It is a linear functional on E, i.e. $I'(u) \in E^*$, the dual space of E. A <u>critical point</u> of I is a point at which $I'(u) = 0$, i.e.

$$I'(u)\phi = 0 \tag{0.1}$$

for all $\phi \in E$. We then call $I(u)$ a <u>critical value</u> of I. In applications to differential equations this situation is of interest since satisfying (0.1) corresponds to obtaining a weak solution of the differential equation. Thus when applicable, critical point theory serves as an existence mechanism for obtaining weak solutions of differential equations as critical points of corresponding functionals.

*Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53705

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The simplest kinds of critical points are local maxima and minima of I. However we are primarily interested in indefinite functionals, i.e. functionals which may not be bounded from above or from below even modulo subspaces or submanifolds of finite dimension or codimension, and such functionals may not possess any local maxima or minima. Thus finding critical points becomes a more subtle question. The various critical points we obtain in this paper have the common feature that they are characterized by a minimax procedure. Such minimax methods not only give us a critical point of I but also estimates for the corresponding critical value that can be useful in some applications.

In §1, several technicalities will be discussed including the so-called Deformation Theorem which plays a role in obtaining critical points. The hypotheses that one needs in a PDE setting to show a functional lies in $C^1(E,R)$ and satisfies appropriate compactness conditions will be described. The basic ideas of minimax theory will also be introduced. The Mountain Pass Theorem and some PDE applications are the main topic in §2. A brief discussion of (Brouwer and Leray-Schauder) degree theory is given in §3 and a Saddle Point Theorem is proved. A generalized version of the Mountain Pass Theorem and some PDE applications are given in §4. The role of symmetries in obtaining multiple critical points of a functional is discussed in §5 and appropriate tools for studying this question, in particular the notion of genus and its properties are introduced. This machinery is used to study constrained variational problems, a theorem of Clark, and applications. In §6 a symmetric version of the Mountain Pass Theorem is given together with an application. Lastly §7 deals with perturbations from symmetry in a PDE setting.

Some other recent sources for material on critical point theory and applications are Nirenberg [1], Berger [2], and Rabinowitz [3], [4]. Some older references are Ljusternik and Schnirelmann [5], Krasnoselski [6], Vainberg [7], Palais [8], Schwartz [9] and Browder [10].

We thank again the members of the Organizing Committee and the participants for their gracious hospitality.

§1. THE DEFORMATION THEOREM AND OTHER TECHNICALITIES

This section treats some technical results that will be important for the sequel. In particular we will describe the Deformation Theorem and will discuss hypotheses under which the functionals we study in the PDE setting are continuously differentiable and satisfy appropriate compactness conditions.

Unless otherwise indicated in this paper, E denotes a real Banach space. Weak convergence in E will be denoted by $\rightharpoonup$ and strong convergence by $\rightarrow$. In order to obtain critical points of $I \in C^1(E,R)$, some "compactness" structure is generally required for I. A useful hypothesis in this direction is the Palais-Smale condition (PS). We say I satisfies (PS) if any sequence (u_m) such that $|I(u_m)|$ is uniformly bounded and $I'(u_m) \to 0$ possesses a convergent subsequence. Several examples of (PS) will be exhibited in the course of our future PDE applications so we will not pause now to give an example.

For $I \in C^1(E,R)$ and $c,s \in R$, let

$$K_c \equiv \{u \in E \mid I(u) = c \text{ and } I'(u) = 0\} \text{ and } A_s \equiv \{u \in E \mid I(u) \le s\} .$$

An important technical result that will be used repeatedly is the following:

<u>Theorem 1.1 (Deformation Theorem):</u> Let $I \in C^1(E,R)$ and satisfy (PS). Then for any $c \in R$, $\bar{\varepsilon} > 0$, and neighborhood $\mathcal{O}$ of K_c, there exists an $\varepsilon \in (0,\bar{\varepsilon})$ and $\eta \in C([0,1] \times E,E)$ such that for all $u \in E$ and $t \in [0,1]$:

1° $\eta(0,u) = u$

2° $\eta(t,u) = u$ if $I(u) \notin [c - \bar{\varepsilon},c + \bar{\varepsilon}]$

3° $\eta(t,\cdot)$ is a homeomorphism of E onto E

4° $\|\eta(t,u) - u\| \le 1$

5° $\eta(1,A_{c+\varepsilon}\setminus\mathcal{O}) \subset A_{c-\varepsilon}$

6° If $K_c = \phi$, $\eta(1,A_{c+\varepsilon}) \subset A_{c-\varepsilon}$

7° If $I(u)$ is even in u, $\eta(t,\cdot)$ is odd in u

Proof: The proof of Theorem 1.1 can be found in [11] or [3]. See also [8] or [10]. We will briefly mention some of the ideas behind the proof. Suppose $E = R^n$ and $I \in C^2$. Then the ordinary differential equation

$$\frac{d\psi}{dt} = -I'(\psi), \qquad \psi(0,u) = u \tag{1.2}$$

possesses a unique solution defined on some maximal t interval $(t^-(u),t^+(u))$ for each $u \in E$. Thus ψ satisfies 1°, 3°, 7° above and where defined we have

$$\frac{d}{dt} I(\psi(t,u)) = I'(\psi) \cdot \frac{d\psi}{dt} = -|I'(\psi)|^2 \tag{1.3}$$

Consequently by (1.3), I strictly decreases along orbits of (1.2) unless u is a critical point of I and this makes 5°-6° seem plausible. Unfortunately $t^+(u)$ may not exceed 1 nor need 2°, 4° be satisfied. However by replacing the right hand side of (1.2) by $-\chi(\cdot)I'(\cdot)$ where χ is an appropriately scaled cut off function, it is not difficult to show the resulting flow satisfies 1°-7°. The general case is technically more difficult since I is merely C^1 and $I' \in E^*$, not E, while $\frac{d\psi}{dt} \in E$ so (1.2) makes no sense. See e.g. the references mentioned above for the details.

Remark 1.4: (i) Related results can be obtained if E is not a linear space but a manifold. See e.g. [8] or [10]. (ii) The requirement that I be C^1 has been weakened somewhat by Chang [12]. (iii) One does not need (PS) for the proof of Theorem 1.1, but only a local version thereof such as: Whenever $I(u_m) \to c$ and $I'(u_m) \to 0$, (u_m) is precompact. (iv) Even weaker versions of (PS) suffice in some situations. See e.g. [13-16]. (v) One is sometimes interested in situations in which ψ corresponds to a positive rather than negative gradient flow. Calling the resulting map $\hat{\eta}$ we still get Theorem 1.1 with some minor modifications. E.g. 6° becomes

$$\hat{\eta}(1,\{u \in E|I(u) > c - \varepsilon\}) \subset \{u \in E|I(u) > c + \varepsilon\}$$

Before beginning with the PDE technicalities, we will discuss the basic ideas behind the use of minimax methods in critical point theory. Suppose $I \in C^1(E,\mathbb{R})$ satisfies (PS) and there exists a family $\mathcal{S}$ of subsets of E which is invariant under a gradient or gradient-like flow (as in (1.2)). Define

$$c \equiv \inf_{A \in \mathcal{S}} \sup_{u \in A} I(u) \tag{1.5}$$

Suppose further that $-\infty < c < \infty$. Then c is a critical value of I. For if not, by Theorem 1.1 with e.g. $\bar{\varepsilon} = 1$, there is an $\varepsilon \in (0,1)$ and $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon} . \tag{1.6}$$

Choose $A \in \mathcal{S}$ such that

$$\sup_A I \le c + \varepsilon . \tag{1.7}$$

By hypothesis, $\eta(1,\cdot) : \mathcal{S} \to \mathcal{S}$ and therefore $\eta(1,A) \in \mathcal{S}$. Hence by (1.6) and (1.7),

$$\sup_{u \in A} I(\eta(1,u)) \le c - \varepsilon$$

contrary to (1.5).

We will see many applications of this argument in the sequel. In practice (assuming we have already verified (PS)), the main difficulties are in finding $\mathcal{S}$ and avoiding critical values that may already be known. As an almost trivial example of this argument, we have

<u>Proposition 1.8:</u> Suppose $I \in C^1(E,\mathbb{R})$, satisfies (PS), and is bounded from below. Then

$$c = \inf_E I$$

is a critical value of I.

<u>Proof:</u> Take $\mathcal{S} = \{\{x\} \mid x \in E\}$. Then $\eta(1,\cdot)$ trivially maps $\mathcal{S} \to \mathcal{S}$ and (1.5) reduces to the $\inf$ of I on E.

We will conclude this section with some results that are necessary for our later PDE applications. We prefer to work with the simplest such situation and therefore will restrict our applications to problems of the form:

$$-\Delta u = p(x,u), \quad x \in \Omega \qquad (1.9)$$
$$u = 0, \qquad x \in \partial\Omega$$

where here and in all of our PDE applications $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary. We assume p satisfies

(p_1) $p \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$,

and

(p_2) there are constants $a_1, a_2 > 0$ such that

$$|p(x,\xi)| \leq a_1|\xi|^s + a_2$$

where $0 \leq s < (n+2)(n-2)^{-1}$ if $n > 2$.

Hypothesis (p_2) is dictated by the Sobolev Embedding Theorem and can be weakened if $n = 2$. Our results also apply when $n = 1$ in which case (p_2) is not needed.

Let P denote the primative of p, i.e.

$$P(x,\xi) = \int_0^\xi p(x,t)dt$$

and let $E = W_0^{1,2}(\Omega)$, the usual Sobolev space obtained as the closure of $C_0^\infty(\Omega)$ under

$$\|u\|_E^2 \equiv \|u\|^2 = \int_\Omega |\nabla u|^2 dx$$

(The usual norm in E is

$$\left(\int_\Omega (|\nabla u|^2 + u^2)dx\right)^{1/2}$$

which by the Poincaré inequality is equivalent to the norm we use). Even if not stated explicitly there, in each of our future PDE

applications, $E = W_0^{1,2}(\Omega)$. Define for $u \in E$,

$$J(u) = \int_\Omega P(x,u(x))dx .$$

Proposition 1.10: Let p satisfy (p_1)-(p_2). Then $J \in C^1(E,\mathbb{R})$ and for $u,\phi \in E$,

$$J'(u)\phi = \int_\Omega p(x,u(x))\phi(x)dx .$$

Moreover J is weakly continuous, i.e. $u_m \rightharpoonup u$ in E implies $J(u_m) \to J(u)$, and $J'(u)$ maps weakly convergent to strongly convergent sequences, i.e. $u_m \rightharpoonup u$ in E implies

$$\int_\Omega p(x,u_m)\phi dx \to \int_\Omega p(x,u)\phi dx$$

uniformly for ϕ in the unit ball in E.

Proof: For the proof of these facts, see [3] or [13].

With these preliminaries, the relationship between (1.9) and critical point theory becomes clear. Set

$$I(u) \equiv \int_\Omega \frac{1}{2} |\nabla u|^2 dx - J(u) = \frac{1}{2} \|u\|^2 - J(u) \tag{1.11}$$

The form of I and Proposition 1.10 imply $I \in C^1(E,\mathbb{R})$ and

$$I'(u)\phi = \int_\Omega (\nabla u \cdot \nabla\phi - p(x,u)\phi)dx \tag{1.12}$$

for all $\phi \in E$. Thus a critical point of I is a weak solution of (1.9).

Proposition 1.13: If p satisfies (p_2) and

(p_1') p is locally Lipschitz continuous in $\bar{\Omega} \times \mathbb{R}$,

any weak solution of (1.9) is a classical solution.

Proof. See e.g. [17].

In order to apply the abstract theorems we will be studying next to (1.11), we must verify that I satisfies (PS). This will be done in later sections under various hypotheses on p. The verification process can be simplified with the aid of the following:

Proposition 1.14: Suppose p satisfies (p_1), (p_2) and I is defined by (1.11). If (u_m) is a sequence in E such that (u_m) is bounded and $I'(u_m) \to 0$, then (u_m) possesses a convergent subsequence.

Remark 1.15: By Proposition 1.14, to verify (PS) in our PDE setting all we need show is whenever $I(u_m)$ is bounded and $I'(u_m) \to 0$, then (u_m) is bounded.

Proof of Proposition 1.14: Since (u_m) is bounded, it possesses a weakly convergent subsequence which we also denote by (u_m). Say $u_m \rightharpoonup u$. Let ℓ denote the canonical injection of E^* to E. The form of I' implies

$$\ell\, I'(u) = u - \ell\, J'(u) \ . \tag{1.16}$$

Thus

$$\ell\, I'(u_m) = u_m - \ell\, J'(u_m) \to 0 \tag{1.17}$$

and by Proposition 1.10, $\ell\, J'(u_m)$ converges to $\ell\, J'(u)$. Hence u_m has a convergent subsequence.

§2. THE MOUNTAIN PASS THEOREM

In this section we will study the Mountain Pass Theorem and some variants of this result. Below $B_R(x)$ denotes the open ball of radius R in E centered at x. If $x = 0$, we simply write B_R.

Theorem 2.1 (Mountain Pass Theorem [18]): Let $I \in C^1(E,\mathbb{R})$ and satisfy (PS). Suppose I also satisfies

(I_1) $I(0) = 0$ and there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} > \alpha$,

(I_2) There is an $e \in E \setminus \bar{B}_\rho$ such that $I(e) < 0$,

Then I possesses a critical value $c > \alpha$ which can be characterized as

$$c \equiv \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)) \tag{2.2}$$

where

$$\Gamma = \{g \in C([0,1],E) \mid g(0) = 0, g(1) = e\}$$

Proof: Since ∂B_ρ separates 0 and e and each curve $g([0,1])$ joins 0 and e, $g([0,1]) \cap \partial B_\rho \neq \phi$ and therefore

$$\max_{t\in[0,1]} I(g(t)) \geq \alpha$$

via (I_1). Hence $c \geq \alpha$. To see that c is a critical value, suppose not. Then we can invoke Theorem 1.1 with $\bar{\varepsilon} = \frac{\alpha}{2}$ and $\mathcal{O} = \phi$. With ε and η as given by Theorem 1.1, choose $g \in \Gamma$ such that

$$\max_{t\in[0,1]} I(g(t)) \leq c + \varepsilon \tag{2.3}$$

and consider $\eta(1,g(t)) \equiv h(t)$. If $h \in \Gamma$, (2.3) and 6° of Theorem 1.1 imply that

$$\max_{t\in[0,1]} I(h(t)) \leq c - \varepsilon \tag{2.4}$$

contrary to the definition of c in (2.2). Thus c must be a critical value of I. To verify that $h \in \Gamma$, observe that $h \in C([0,1],E)$. Moreover our choice of $\bar{\varepsilon}$ and 2° of Theorem 1.1 imply that $h(0) = \eta(1,g(0)) = \eta(1,0) = 0$ and $h(1) = \eta(1,g(1)) = \eta(1,e) = e$. Thus $h \in \Gamma$ and the proof is complete.

Remark 2.5: The Mountain Pass Theorem is due to Ambrosetti and Rabinowitz [18]. It is so named because (I_1)-(I_2) imply 0 and e are separated by a "mountain range" and therefore there must be a mountain pass through this range. Thus our characterization of c seems natural. However there are other ways to characterize a critical value of I which are also geometrically natural and which may in general give a different critical value than c above. Indeed there is no definitive characterization of a critical value of I for the class of problems we study. Choosing sets with respect to which to minimax I is a very ad hoc process.

The next result illustrates the above remarks by producing another critical value for I which may not equal c.

Theorem 2.6 [18]: Under the hypotheses of Theorem 2.1, I possesses a critical value b such that $\alpha \leq b \leq c$ where b can be characterized as

$$b = \sup_{B \in W} \inf_{u \in \partial B} I(u) \tag{2.7}$$

where

$$W = \{B \subset E \mid 0 \in B \text{ open and } e \notin \bar{B}\} .$$

<u>Proof:</u> By (I_1),

$$\inf_{\partial B_\rho} I > \alpha$$

so $b > \alpha$. If $B \in W$ and $g \in \Gamma$,

$$B \cap g([0,1]) \neq \phi .$$

Thus if w lies in this intersection,

$$\inf_B I \leq I(w) \leq \max_{g([0,1])} I .$$

Since this is true for any $B \in W$ and $g \in \Gamma$, $b \leq c$. Lastly if B is not a critical value of I, letting $\bar{\varepsilon} = \frac{\alpha}{2}$, Theorem 1.1 and Remark 1.4 (v) give us $\varepsilon \in (0,\bar{\varepsilon})$ and $\hat{\eta} \in C([0,1] \times E,E)$ such that

$$\hat{\eta}(1,\hat{A}_{b-\varepsilon}) \subset \hat{A}_{b+\varepsilon} \tag{2.8}$$

where $\hat{A}_s = \{u \in E \mid I(u) > s\}$. Choose $B \in W$ such that

$$\inf_{\partial B} I > b - \varepsilon . \tag{2.9}$$

and consider $\hat{\eta}(1,B)$. This is an open set since $\hat{\eta}(1,\cdot)$ is a homeomorphism. Moreover $\hat{\eta}(1,0) = 0$ and $\hat{\eta}(1,e) = e$ via our choice of $\bar{\varepsilon}$ so $0 \in \hat{\eta}(1,B)$ and $e \notin \overline{\eta(1,B)}$. Thus $\hat{\eta}(1,B) \in W$ and

$$\inf_{\partial\hat{\eta}(1,B)} I \leq b . \tag{2.10}$$

But $\partial\hat{\eta}(1,B) = \hat{\eta}(1,\partial B)$ since $\hat{\eta}(1,\cdot)$ is a homeomorphism and therefore by (2.8) and (2.9),

$$\inf_{\partial\hat{\eta}(1,B)} I > b + \varepsilon \tag{2.11}$$

contrary to (2.10).

Exercise 2.12: Give an example where $b \neq c$ (e.g. in $\mathbb{R}$)

Using the dual approach to the Mountain Pass Theorem as in this last result, we can prove a critical point theorem for a "degenerate" situation (see also [19]).

Theorem 2.13: Let $I \in C^1(E,\mathbb{R})$ and satisfy (PS). Suppose I satisfies

(I_1') $I(0) = 0$ and there is a $\rho > 0$ such that $I|_{\partial B_\rho} > 0$

and (I_2). Then I possesses a critical value $b > 0$, as characterized by (2.7). Moreover if $b = 0$, there exists a critical value of I on ∂B_ρ.

Proof: If $b > 0$, the proof of Theorem 2.6 carries over to this case without any change. Thus suppose $b = 0$. Without loss of generality we can assume

$$\min(\rho, \|e\| - \rho) > 1 . \tag{2.14}$$

If I has a critical value on ∂B_ρ, we are through. Thus suppose $I' \neq 0$ on ∂B_ρ. Therefore since K_0 is compact by (PS), there is a neighborhood $\mathcal{O}$ of K_0 such that $\mathcal{O} \cap \partial B_\rho = \phi$. By Theorem 1.1 with $\bar{\varepsilon} = 1$, there is an $\varepsilon \in (0,1)$ and $\hat{\eta} \in C([0,1] \times E, E]$ such that

$$\hat{\eta}(1, \hat{A}_{-\varepsilon} \backslash \mathcal{O}) \subset \hat{A}_\varepsilon .$$

In particular $\hat{\eta}(1, \partial B_\rho) = \partial \hat{\eta}(1, B_\rho) \subset \hat{A}_\varepsilon$ and

$$\inf_{\partial \hat{\eta}(1,B_\rho)} I > \varepsilon .$$

Thus we have a contradiction to $b = 0$ provided that $\hat{\eta}(1,B_\rho) \in W$. It suffices to verify that $0 \in \hat{\eta}(1,B_\rho)$ and $e \notin \overline{\hat{\eta}(1,B_\rho)}$. Since $\hat{\eta}(1,\cdot)$ is a homeomorphism of E onto E, there exists an $x \in E$ such that $\hat{\eta}(1,x) = 0$. Moreover by 4° of Theorem 1.1, $\|\hat{\eta}(1,x) - x\| = \|x\| < 1 < \rho$ via (2.14) so $x \in B_\rho$. Similarly there is a $y \in E$ such that $\hat{\eta}(1,y) = e$. If $y \in \overline{B}_\rho$, $\|y - \hat{\eta}(1,y)\| = \|y - e\| < 1$ contrary to (2.14). Thus I has a critical point on ∂B_ρ and the proof is complete.

Remark 2.15: The conclusions of Theorem 2.13 hold if in (I_1') we replace ∂B_ρ by ∂B for some $B \in W$. Indeed the same proof works.

Corollary 2.16 [3]: Let $I \in C^1(E,R)$ and satisfy (PS). If I possesses a pair of local minima, then I possesses a third critical point.

Proof: Suppose the local minima occur at x_1, x_2 respectively, $I(x_i) = \alpha_i$, $i = 1,2$, and $\alpha_1 > \alpha_2$. Without loss of generality we can assume $\alpha_1 = 0$ and $x_1 = 0$. Since 0 is a local minimum for I, (I_1') is satisfied and the result follows from Theorem 2.13.

Remark 2.17: If in the setting of Corollary 2.16, b as defined by (2.7) equals 0, then K_0 contains a component which meets ∂B_r for each small r. For since 0 is a local minimum for I, there is an $r > 0$ such that $I > 0$ in B_r. Thus by Remark 2.15 for any $B \in W$ with $B \subset \overline{B_r}$, $K_0 \cap \partial B \neq \phi$. The result then follows from a standard argument in point set topology (see e.g. [20]).

Now we turn to some applications of the Mountain Pass Theorem to PDE's. Consider (1.9).

Theorem 2.18 [18]: If p satisfies (p_1)-(p_2),

(p_3) $p(x,\xi) = o(|\xi|)$ as $\xi \to 0$,

(p_4) There is a $\mu > 2$ and $r > 0$ such that

$$0 < \mu P(x,\xi) \leq \xi p(x,\xi) \quad \text{for} \quad |\xi| \geq r ,$$

then (1.9) possesses a nontrivial weak solution.

Remark 2.19: Note that (p_3) implies that $u \equiv 0$ is a solution of (1.9) which we will call the trivial solution of (1.9). Hypothesis (p_4) implies there are constants $a_3, a_4 > 0$ such that

$$P(x,\xi) \geq a_3|\xi|^{\mu} - a_4 \tag{2.20}$$

for all $\xi \in R$. Thus P grows at a "superquadratic" rate and by (p_4), p at a "superlinear" rate as $|\xi| \to \infty$. Consequently by (1.11) and (2.20),

$$I(t,u) \leq \frac{t^2}{2} \|u\|^2 - \int_\Omega (a_3|u|^{\mu} - a_4)dx \to -\infty \tag{2.21}$$

as $t \to \infty$ for any $u \in E\backslash\{0\}$ which shows that I is not bounded from below. Moreover I is not bounded from above. Indeed choose any orthornormal basis (e_m) for $E = W_0^{1,2}(\Omega)$. Then for any $R > 0$,

$Re_m \to 0$ and $J(Re_m) \to 0$ via Proposition 1.10. Hence

$$I(Re_m) > \frac{1}{4} R^2$$

for all large $m = m(R)$. Since R is arbitrary, I is not bounded from above in E.

Proof of Theorem 2.18: We will show that I as defined by (1.11) satisfies the hypotheses of Theorem 1.1. Clearly $I(0) = 0$ and hypotheses (p_1)-(p_2) and Proposition 1.10 show $I \in C^1(E,\mathbb{R})$. Hypothesis (p_4) implies (I_2) via Remark 2.19. The form of I shows (I_1) holds if $J(u) = o(\|u\|^2)$ as $\|u\| \to 0$. By (p_3), for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|\xi| < \delta$ implies $|p(x,\xi)| < \varepsilon|\xi|$ and

$$|P(x,\xi)| < \frac{1}{2}\varepsilon|\xi|^2 . \qquad (2.22)$$

By (p_2), there is an $A_\varepsilon > 0$ such that

$$|P(x,\xi)| < A_\varepsilon|\xi|^{s+1} \qquad (2.22')$$

for $|\xi| > \delta$. Adding these two equations yields

$$|P(x,\xi)| < \frac{\varepsilon}{2}|\xi|^2 + A_\varepsilon|\xi|^{s+1} \qquad (2.23)$$

for all $\xi \in \mathbb{R}$. The Poincaré and Sobolev inequalities then imply

$$|J(u)| < \frac{\varepsilon}{2}\|u\|^2_{L^2} + A_\varepsilon\|u\|^{s+1}_{L^{s+1}} < a_5\left(\frac{\varepsilon}{2} + A_\varepsilon\|u\|^{s-1}\right)\|u\|^2 \qquad (2.24)$$

for all $u \in E$. Since ε is arbitrary, $J(u) = o(\|u\|^2)$ as $\|u\| \to 0$ and (I_1) is satisfied. Lastly suppose (u_m) is a sequence in E such that $|I(u_m)| < M$ and $I'(u_m) \to 0$. Then for all large m and $u \equiv u_m$,

$$M + \beta\|u\| > I(u) - \beta I'(u)u = \qquad (2.25)$$

$$= \left(\frac{1}{2} - \beta\right)\|u\|^2 + \int_\Omega \left(\beta p(x,u)u - P(x,u)\right)dx$$

where $\beta \in \left(0,\frac{1}{2}\right)$. Choosing $\beta = \frac{1}{\mu}$, (p_4) and (2.25) show

$$M_1 + \beta\|u_m\| \geq (\frac{1}{2} - \beta)\|u_m\|^2$$

where M_1 is independent of m. It follows that (u_m) is bounded in E. Therefore by Proposition 1.14, (PS) is satisfied. Hence Theorem 2.18 follows from the Mountain Pass Theorem.

<u>Corollary 2.27 [18]</u>: Suppose p satisfies (p_1'), (p_2), (p_3) and

(p_4') There exists $\mu > 2$ and $r > 0$ such that

$$0 < \mu P(x,\xi) \leq \xi p(x,\xi) \quad \text{for} \quad \xi \geq r .$$

Then (1.9) possesses a classical solution $u > 0$ in Ω.

<u>Proof:</u> Define $\bar{p}(x,\xi) = p(x,\xi)$ if $\xi > 0$ and $\bar{p}(x,\xi) = 0$ if $\xi \leq 0$. If $\bar{P}$ denotes the primative of $\bar{p}$, $\mu\bar{P}(x,\xi) \leq \xi\bar{p}(x,\xi)$ for all $|\xi| \geq r$. The arguments of Theorem 2.18 show

$$\bar{I}(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \bar{P}(x,u)dx$$

satisfies the hypotheses of the Mountain Pass Theorem. Indeed only the proof that $\bar{I}$ satisfies (I_2) need be modified. The verification of (I_2) follows from (2.20) which now holds for $\xi > 0$. Taking e to be a positive function in Ω then easily yields (I_2). Therefore by Theorem 2.1, the equation

$$-\Delta u = \bar{p}(x,u), \quad x \in \Omega \tag{2.28}$$
$$u = 0 \quad , \quad x \in \partial\Omega$$

has a nontrivial weak solution u and by Proposition 1.13 u is a classical solution of (2.28). Consider $\mathcal{D} = \{x \in \Omega | u(x) < 0\}$. Then by the definition of $\bar{p}$,

$$-\Delta u = 0, \quad x \in \mathcal{D} \tag{2.29}$$
$$u = 0, \quad x \in \partial\mathcal{D} .$$

The Maximum Principle then implies $\mathcal{D} = \phi$. Therefore $u \geq 0$ in Ω and hence satisfies (1.9). Since we can write (2.28) as

$$-\Delta u - (\frac{\bar{p}(x,u)^-}{u})u = (\frac{\bar{p}(x,u)^+}{u})u$$

where $\bar{p}^{\pm}$ are respectively the positive and negative parts of $\bar{p}$

and $\bar{p}(x,u)^{\pm}u^{-1}$ are continuous via (p_3), the Strong Maximum Principle implies $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$ where $\nu(x)$ is the outward pointing normal to $\partial\Omega$.

Corollary 2.30: If p satisfies (p_1'), (p_2)-(p_4), (1.9) possesses a classical solution $u > 0$ and a classical solution $w < 0$ in Ω

Proof: Since (p_4) implies (p_4'), the existence of u follows from Corollary 2.27. A truncation argument similar to that of Corollary 2.27 also yields the existence of the negative solution w.

Remrk 2.31: More careful arguments avoiding the classical Maximum Principle give weak solutions u and w as in Corollary 2.30 with (p_1') replaced by (p_1). An interesting open question is whether one can get the positive and negative solutions of (1.9) in a more direct fashion without using the truncation arguments of the above Corollaries.

Remark 2.32: An identity of Pohozaev [21] for solutions of (1.9) when p is independent of x says

$$2n \int_{\Omega} P(u)dx + (2 - n) \int_{\Omega} p(u)udx = \int_{\partial\Omega} x\cdot\nu(x)|\nabla u|^2 dS \ . \tag{2.33}$$

Thus if Ω is starshaped with respect to the origin, $x\cdot\nu(x) > 0$ and (2.33) implies

$$\int_{\Omega} P(u)dx > \frac{n-2}{2n} \int_{\Omega} p(x,u)udx \tag{2.34}$$

Thus if $P(u) = (s + 1)^{-1}|u|^{s+1}$, (2.34) shows $s < (n + 2)(n - 2)^{-1}$ and a growth condition is necessary to get nontrivial solutions of (1.9). An interesting open question is to better understand the relationship between the geometry of the domain and the growth rate of $p(x,\xi)$. E.g. for domains with holes, there may exist nontrivial solutions even if $s > (n + 2)(n - 2)^{-1}$. This is known in particular for p a pure power in an annular domain in $\mathbb{R}^n$.

For our next PDE application, consider the nonlinear eigenvalue problem

$$-\Delta u = \lambda p(x,u), \quad x \in \Omega \tag{2.35}$$
$$u = 0 \quad , \quad x \in \partial\Omega$$

where $\lambda \in \mathbb{R}$.

<u>Theorem 2.36 [18]:</u> Suppose p satisfies (p_1'), (p_3), and

(p_5) There is an $r > 0$ such that $p(\xi) > 0$ in $(0,r)$ and $p(r) = 0$.

Then there exists a $\bar{\lambda} > 0$ such that for all $\lambda > \bar{\lambda}$, (2.35) has at least two classical solutions which are positive in Ω.

<u>Proof:</u> Define $\bar{p}(\xi) = p(\xi)$ for $\xi \in [0,r]$ and $\bar{p}(\xi) = 0$ otherwise. Then $\bar{p}$ satisfies (p_1'), (p_3), and (p_5). If u is a solution of

$$-\Delta u = \lambda \bar{p}(u), \quad x \in \Omega \tag{2.37}$$
$$u = 0 \quad , \quad x \in \partial\Omega$$

the argument of Corollary 2.27 shows $\mathcal{D}$ (as defined there) $= \phi$ and $u > 0$ in Ω. A similar argument shows $\mathcal{D}^+ = \{u \in \Omega | u > r\} = \phi$ Hence $0 < u < r$ in Ω and u satisfies (2.35). Thus to prove the theorem, by these observations and Proposition 1.13 it suffices to find nontrivial critical points of

$$I_\lambda(u) \equiv \frac{1}{2} \|u\|^2 - \lambda \int_\Omega \bar{P}(u)dx \tag{2.38}$$

on $E = W_0^{1,2}(\Omega)$ where $\bar{P}$ is the primative of $\bar{p}$. Since $\bar{p}$ satisfies (p_2) with $s = 0$, $I_\lambda \in C^1(E,\mathbb{R})$ and I_λ satisfies (I_1) via (p_2), (p_3) as in Theorem 2.8. Suppose (u_m) is a sequence such that $I_\lambda(u_m) \le M$. Then by (p_2) with $s = 0$,

$$\frac{1}{2} \|u_m\|^2 - a_1 \int_\Omega |u_m| dx - a_5 \le I_\lambda(u_m) \le M \, . \tag{2.39}$$

Hence by the Hölder and Poincaré inequalities (u_m) is bounded and (PS) follows from Proposition 1.14.

Inequality (2.39) also shows I_λ is bounded from below. Thus

$$b_\lambda \equiv \inf_E I_\lambda$$

is a critical value of I_λ by Proposition 1.8. It is possible that $b_\lambda = 0$ and corresponds to the trivial solution $u \equiv 0$ of (2.35) (Indeed $b_\lambda = 0$ for small λ). However let $\phi \in E\backslash\{0\}$ such that $\phi > 0$ and $\phi(x) \in [0,r)$ for $x \in \Omega$. Then for λ sufficiently large, $I_\lambda(\phi) < 0$ and therefore $b_\lambda < 0$. Set $\bar{\lambda} = \inf\{\lambda \in \mathbb{R} | b_\lambda < 0\}$. Then for all $\lambda > \bar{\lambda}$, I_λ has a critical point u_λ^- such that $I_\lambda(u_\lambda^-) = b_\lambda < 0$ and u_λ^- is a positive solution of (2.35). Moreover since $I_\lambda(u_\lambda^-) < 0$, I_λ satisfies (I_2). Hence by the Mountain Pass Theorem, I_λ has a second critical point u_λ^+ such that $I_\lambda(u_\lambda^+) > 0$ and u_λ^+ is also a positive solution of (2.35). Thus we have two distinct positive solutions of (2.35) for $\lambda > \bar{\lambda}$ and the proof is complete.

§3. THE SADDLE POINT THEOREM

Our goal in this section is to prove:

Theorem 3.1 (Saddle Point Theorem) [22]: Let $E = V \oplus X$ where V is finite dimensional. Suppose $I \in C^1(E,\mathbb{R})$ and satisfies (PS). If there are constants α and β and a bounded neighborhood D of 0 in V such that

(I_3) $I|_{\partial D} \le \alpha$,

and

(I_4) $I|_X \ge \beta > \alpha$,

then I possesses a critical value $c \ge \beta$. Moreover c can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in D} I(h(u)) \qquad (3.2)$$

where

$$\Gamma = \{h \in C(\bar{D},E) | h(u) = u \text{ for } u \in \partial D\}$$

Remark 3.3: Note that unlike the applications of the Mountain Pass Theorem in §2, there is no known critical point to begin with in Theorem 3.1. If $I(u) \to -\infty$ as $u \to \infty$ in V and $I(u) \to \infty$ as $u \to \infty$ in X, then I satisfies (I_3)-(I_4). This will be the case if I is e.g. convex in X, concave in V, and appropriately coercive.

One of the tools that goes into the proof of Theorem 3.1 and some later results is the theory of topological degree of Brouwer in the finite dimensional case and of Leray and Schauder in the infinite dimensional setting. Therefore we will make a brief digression to discuss the results we will need.

Let $\mathcal{O} \subset \mathbb{R}^n$ be bounded and open, $\phi \in C^1(\mathcal{O},\mathbb{R}^n)$, and $b \in \mathbb{R}^n\backslash\phi(\partial\mathcal{O})$. Consider the equation

$$\phi(x) = b \; . \tag{3.4}$$

We are interested in whether there are any solutions of this equation and if so how many. Suppose $\phi'(x)$ is nonsingular whenever $\phi(x) = b$. Then by the Inverse Function Theorem, solutions of (3.4) are isolated and therefore there can only be finitely many of them since by hypotheses $b \notin \phi(\partial\mathcal{O})$. For this "nice" case we define the (Brouwer) degree of ϕ with respect to $\mathcal{O}$ and b, $d(\phi,\mathcal{O},b)$, to be

$$d(\phi,\mathcal{O},b) \equiv \sum_{x\in\phi^{-1}(b)} \operatorname{sign}|\phi'(x)| \tag{3.5}$$

where $|\phi'(x)|$ denotes the determinant of ϕ' at x. Then $d(\phi,\mathcal{O},b)$ possesses the following properties

1° $d(id,\mathcal{O},b) = 1$ if $b \in \mathcal{O}$; and $= 0$ otherwise

2° $d(\phi,\mathcal{O},b) \neq 0$ implies there exists $x \in \mathcal{O}$ such that $\phi(x) = b$

3° $d(\phi,\mathcal{O},b) = 0$ if $b \notin \phi(\mathcal{O})$

4° (Continuity of d in ϕ): $d(\psi,\mathcal{O},b) = d(\phi,\mathcal{O},b)$ for all ψ near ϕ

5° (Additivity) If $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ where $\mathcal{O}_1 \cap \mathcal{O}_2 = \phi$ and $b \notin \phi(\partial\mathcal{O}_1) \cup \phi(\partial\mathcal{O}_2)$, $d(\phi,\mathcal{O},b) = d(\phi,\mathcal{O}_1,b) + d(\phi,\mathcal{O}_2,b)$

In 1°, id denotes the identity map. The proofs of the above statements are immediate from the definition with the exception of 4° (which refers to "nice" C^1 mappings ψ near ϕ) which follows with the aid of the Inverse Function Theorem - see [9].

This notion of degree extends from the class of nice C^1 ϕ's to $C(\bar{\mathcal{O}},\mathbb{R}^n)$. See [9] for the proof of:

Theorem 3.7: There is an integer valued map $d = d(\phi,\mathcal{O},b)$ defined for all bounded open sets $\mathcal{O} \subset R^n$, $\phi \in C(\bar{\mathcal{O}},R^n)$, and $b \in R^n\backslash\phi(\partial\mathcal{O})$ and which satisfies 1°-5° of (3.6). Moreover d is given by (3.5) for "nice" ϕ.

Remark 3.8: In Theorem 3.7, 4° of (3.6) refers to all maps ψ near ϕ in $C(\bar{\mathcal{O}},R^n)$.

An important consequence of 4° is the homotopy invariance property of d.

Proposition 3.9: If $H \in C([0,1] \times \bar{\mathcal{O}},R^n)$ and $b \notin H([0,1] \times \partial\mathcal{O})$, then $d(H(t,\cdot),\mathcal{O},b)$ is independent of t.

Proof: By 4° of (3.6), $d(H(t,\cdot),\mathcal{O},b)$ is continuous in t and is integer valued; hence the result.

Proposition 3.9 implies that $d(\phi,\mathcal{O},b)$ depends only on the values of ϕ on $\partial\mathcal{O}$:

Corollary 3.10: If $\phi,\psi \in C(\bar{\mathcal{O}},R^n)$, $\phi = \psi$ on $\partial\mathcal{O}$, and $b \notin R^n\backslash\phi(\partial\mathcal{O})$, then $d(\phi,\mathcal{O},b) = d(\psi,\mathcal{O},b)$.

Proof: Set $H(t,x) = t\phi(x) + (1 - t)\psi(x)$ and invoke Proposition 3.9.

The finite dimensional degree theory that has just been described has an extension that is valid in an infinite dimensional setting. Let E be a real Banach space and $\Phi \in C(\bar{\mathcal{O}},E)$ where $\mathcal{O} \subset E$ is bounded and open and $\Phi(u) = u - T(u)$ where T is compact. The resulting degree theory is due to Leray and Schauder and can be obtained from Theorem 3.7 by a limit process. See [9] or [23] for details. For later reference we state:

Theorem 3.11: Let E be a real Banach space. There exists an integer valued map $d = d(\Phi,\mathcal{O},b)$ defined for all bounded open sets $\mathcal{O} \subset E$, $\Phi(u) = u - T(u) \in C(\bar{\mathcal{O}},E)$ where T is compact, and $b \in E\backslash\Phi(\partial\mathcal{O})$ and which satisfies 1°-5° of (3.6).

Remark 3.12: It follows from their proofs that Leray-Schauder degree also satisfies the conclusions of Proposition 3.9 and Corollary 3.10.

Now we turn to the

Proof of Theorem 3.1: Let P denote the projector of E onto V obtained from the given splitting of E. If $h \in \Gamma$, then $Ph \in C(\bar{D},V)$ and we can identify V with R^n for some n.

Moreover for $u \in \partial\mathcal{D}$, $Ph(u) = Pu = u \neq 0$. Thus $d(Ph,\mathcal{D},0)$ is defined and by Corollary 3.10 and 1° of Theorem 3.7,

$$d(Ph,\mathcal{D},0) = d(id,\mathcal{D},0) = 1$$

Thus there exists $x \in \mathcal{D}$ such that $Ph(x) = 0$. Since $h(x) = (id - P)h(x) \in X$, by (I_4)

$$\max_{\mathcal{D}} I(h(u)) \geq I(h(x)) \geq \beta \,. \tag{3.13}$$

But (3.13) holds for each $h \in \Gamma$ so $c \geq \beta$. If c is not a critical value of I, set $\bar{\varepsilon} = \frac{1}{2}(\beta - \alpha)$ and invoke the Deformation Theorem to obtain ε and η as usual. Choose $h \in \Gamma$ so that

$$\max_{\mathcal{D}} I(h(u)) \leq c + \varepsilon \tag{3.14}$$

and consider $\eta(1,h)$. The choice of $\bar{\varepsilon}$ implies $\eta(1,h(u)) = u$ if $u \in \partial\mathcal{D}$ and therefore $\eta(1,h) \in \Gamma$. But then

$$\max I(\eta(1,h(u))) \leq c - \varepsilon \tag{3.15}$$

contrary to (3.2).

For some applications of Theorem 3.1, see [22].

§4. A GENERALIZED MOUNTAIN PASS THEOREM

There are many variations of the Mountain Pass Theorem, some of which essentially contain both Theorems 2.1 and 3.1. See e.g. [1], [13], [14]. In this section a relatively simple extension of Theorem 2.1 which requires the degree theory machinery of §3 will be proved and a PDE application will be given.

The main abstract result in this section is

Theorem 4.1 [24]: Let $E = V \oplus X$ where V is finite dimensional and let $I \in C^1(E,\mathbb{R})$ and satisfy (PS). Suppose further I satisfies

(I_5) There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$ and

(I_6) There is an $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q \equiv (\bar{B}_R \cap V) \oplus \{re \mid 0 < r < R\}$, $I|_{\partial Q} \leq 0$.

Then I possesses a critical value $c \geq \alpha$ which can be

characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in Q} I(h(u)) \tag{4.2}$$

where

$$\Gamma = \{h \in C(Q,E) \mid h(u) = u \ \text{ if } \ u \in \partial Q\} .$$

Remark 4.3: (I_6) is satisfied if $I|_V \le 0$ and there is an $e \in \partial B_1 \cap X$ and $\bar{R} > \rho$ such that $I(u) \le 0$ if $u \in \mathrm{span}\{V,e\}$ and $\|u\| > \bar{R}$. If $I(0) = 0 > I(e)$, $V = \{0\}$, and $X = E$, we are back in the setting of Theorem 2.1.

Proof of Theorem 4.1: Suppose $c > \alpha$, where c is defined by (4.2). Then a familiar argument completes the proof: If c is not a critical value of I, by the Deformation Theorem with $\bar{\varepsilon} = \frac{\alpha}{2}$, there exists ε and η as usual. Choose $h \in \Gamma$ such that

$$\max_{u\in Q} I(h(u)) \le c + \varepsilon \tag{4.4}$$

and consider $\eta(1,h)$. Clearly $\eta(1,h) \in C(Q,E)$ and by our choice of $\alpha, \eta(1,h(u)) = u$ on ∂Q. Thus $\eta(1,h) \in \Gamma$ but by (4.4),

$$\max_{u\in Q} I(\eta(1,h(u))) \le c - \varepsilon , \tag{4.5}$$

contrary to (4.2).

Thus the only novelty in the proof is to show $c > \alpha$. It suffices to prove

$$h(Q) \cap \partial B_\rho \cap X \neq \phi \tag{4.6}$$

for each $h \in \Gamma$ for then if $h \in \Gamma$ and $w \in Q$ such that $h(w) \in \partial B_\rho \cap X$,

$$\max_{u\in Q} I(h(u)) \ge I(w) \ge \inf_{\partial B_\rho \cap X} I \tag{4.7}$$

Since h is arbitrary, (4.7) and (I_5) imply $c \ge \alpha$.

To verify (4.6), let P denote the projector of E onto V given by our splitting of E. Then (4.6) is equivalent to

$$Ph(u) = 0, \quad \|(id - P)h(u)\| = \rho \tag{4.8}$$

for some $u \in Q$, u depending on h. If $u \in Q$, $u = v + re$ where $v \in \bar{B}_R \cap V$ and $0 < r < R$. Define

$$\Phi(r,v) \equiv (\|(id - P)h(v + re)\|, Ph(v + re))$$

Thus Φ is continuous on $\mathbb{R} \times V$ (which we can identify with $\mathbb{R} \times \mathbb{R}^k$ for some k). Note that on ∂Q, $h = id$ so for $u \in \partial Q$,

$$\Phi(r,v) = (\|(id - P)(v + re)\|, P(v + re)) = (r,v) ,$$

i.e. $\Phi = id$ on ∂Q. In particular $\Phi(r,v) \neq (\rho,0)$ on ∂Q and $d(\Phi, \text{int } Q, (\rho,0))$ is defined where int Q denotes the interior of Q. Furthermore

$$d(\Phi, \text{int } Q, (\rho,0)) = d(id, \text{int } Q, (\rho,0)) = 1 \qquad (4.9)$$

by Corollary 3.10 and 1° of (3.6). But (4.9) implies (4.8) has a solution in Q via 2° of (3.6). The proof is complete.

Remark 4.10: An interesting open question in the settings of Theorems 2.1, 3.1, or 4.1 is the following. Suppose $I'(u) = u - T(u)$ where T is compact. Let w be a critical point of I with critical value c where c is given by (2.2), (3.2) or (4.2). If further w is an isolated zero of I', $d(I', B_r(w), 0)$ is defined for small r and is independent of r by 5° of Theorem 3.11. One can therefore ask: What is this degree? Such information would assist in obtaining further zeroes of I'.

Next we will give a PDE application of Theorem 4.1. Consider

$$-\Delta u = \lambda u + p(x,u), \quad x \in \Omega \qquad (4.11)$$
$$u = 0 \qquad , \quad x \in \partial\Omega$$

where Ω is as in (1.9) and p satisfies (p_1)-(p_4) of Theorem 2.18 as well as

(p_6) $\xi p(x,\xi) \geq 0$

for all $\xi \in \mathbb{R}$.

Theorem 4.12 [24]: If p satisfies (p_1)-(p_4) and (p_6), then for each $\lambda \in \mathbb{R}$, (4.11) possesses a nontrivial weak solution.

Proof: Let (λ_k) denote the eigenvalues of

$$-\Delta v = \mu v, \quad x \in \Omega \qquad (4.13)$$
$$v = 0 , \quad x \in \partial\Omega$$

We order the λ_k by increasing magnitude, each eigenvalue being listed a number of times equal to its multiplicity. As is well known, this multiplicity is finite and $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \to \infty$ as $k \to \infty$. Let (v_k) be a corresponding orthonormal sequence of eigenfunctions of (4.13).

If $\lambda < \lambda_1$, the proof of Theorem 2.18 carries over unchanged to the present case and (p_6) is not needed provided that we take as equivalent norm in E

$$\left(\int_\Omega (|\nabla u|^2 - \lambda u^2)dx\right)^{1/2} .$$

Thus the interesting case is $\lambda \geq \lambda_1$. Suppose $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \geq 1$. Set $V = \text{span}\{v_1, \ldots, v_k\}$ and $X = V^\perp$, the orthogonal complement of V. Using e.g. an eigenfunction expansion, it is easy to see that for $u \in X$,

$$\int_\Omega (|\nabla u|^2 - \lambda u^2)dx \geq \alpha_k \|u\|^2 \tag{4.14}$$

where $\alpha_k = 1 - \lambda\lambda_{k+1}^{-1}$. Let

$$I(u) \equiv \int_\Omega \frac{1}{2} (|\nabla u|^2 - \lambda u^2)\, dx - J(u)$$

where J is as in Theorem 2.18. Then as earlier $J(u) = o(\|u\|^2)$ as $u \to 0$ which with (4.14) yields (I_5). Next note that for $u \in V$,

$$\int_\Omega (|\nabla u|^2 - \lambda u^2)dx \leq 0$$

so by (p_6), $I(u) < 0$ for $u \in V$. Moreover for $u \in \tilde{E} \equiv \text{span}\{V, v_{k+1}\}$, by (p_4) and (2.20),

$$I(u) \leq \int_\Omega \left[\frac{1}{2} (|\nabla u|^2 - \lambda u^2) - a_3|u|^\mu + a_4\right]dx \tag{4.15}$$

Since $\hat{E}$ is finite dimensional, the μ term in (4.15) dominates for large u and therefore there exists $\bar{R} > \rho$ such that $I(u) \leq 0$ in $\hat{E}\setminus B_{\bar{R}}$. Thus (I_6) is satisfied with $e = v_{k+1}$ via Remark 4.3.

Once (PS) has been established, Theorem 4.12 follows from Theorem 4.1. To verify (PS), a slight variation of the argument of Theorem 2.18 is required. Suppose $|I(u_m)| \le M$ and $I'(u_m) \to 0$. Let $\beta \in (\frac{1}{2}, \frac{1}{\mu})$. Then for all large m and $u = u_m$ we have

$$M + \beta\|u\| \ge I(u) - \beta I'(u)u = \tag{4.16}$$

$$= \int_\Omega [(\frac{1}{2} - \beta)|\nabla u|^2 - \lambda(\frac{1}{2} - \beta)u^2 - P(x,u) + \beta p(x,u)u]dx$$

$$\ge (\frac{1}{2} - \beta)\|u\|^2 - \lambda(\frac{1}{2} - \beta)\int_\Omega u^2 dx + (\beta\mu - 1)a_3 \int_\Omega |u|^\mu dx - a_5$$

via (p_4) and (2.20). Since by the Hölder and Young inequalities

$$\|u\|_{L^2} \le a_6\|u\|_{L^\mu} \le K(\varepsilon) + \varepsilon\|u\|^\mu_{L^\mu} \tag{4.17}$$

for all $\varepsilon > 0$ where $K(\varepsilon) \to \infty$ as $\varepsilon \to 0$, by choosing ε sufficiently small, the $\|u\|_{L^2}$ term in (4.16) can be absorbed by the $\|u\|_{L^\mu}$ term modulo increasing a_5. It then follows that (u_m) is bounded in E and (PS) is a consequence of Proposition 1.13. The proof is complete.

Remark 4.18: By following the arguments of Corollary 2.30 it is not difficult to show that if p also satisfies (p_1'), (4.11) has a positive and a negative solution for $\lambda \in (-\infty, \lambda_1)$. However if $\lambda > \lambda_1$, this is no longer the case. Indeed suppose $u > 0$ in Ω is a solution of (4.11). As is well known, v_1 has one sign in Ω which we can take to be positive. Then by (4.11) and (4.13),

$$\int_\Omega (-\Delta u)v_1 dx = \int_\Omega u(-\Delta v_1)dx = \lambda_1 \int_\Omega uv_1 dx \tag{4.19}$$

$$= \int_\Omega (\lambda u + p(x,u))v_1 dx$$

or

$$(\lambda_1 - \lambda)\int_\Omega uv_1 dx = \int_\Omega p(x,u)v_1 dx \tag{4.20}$$

Since the right hand side of (4.20) is nonnegative via (p_6) while the left hand side is negative, this situation is not possible.

§5 SYMMETRIES AND MULTIPLE CRITICAL POINTS

When a functional is invariant under a group of symmetries, many situations are known in which it possesses multiple critical points. To minimize technicalities we will treat the simplest such case when the group is $\mathbb{Z}_2$. In particular we will study an even functional, i.e. $I(u) = I(-u)$. (I possesses a $\mathbb{Z}_2$ symmetry since it is invariant under the maps $\{id, -id\}$). The first multiple critical point theorem we know of is a classical result due to Ljusternik [25].

Theorem 5.1: Let $I \in C^1(\mathbb{R}^n, \mathbb{R})$ with I even. Then $I|_{S^{n-1}}$ possesses at least n distinct pairs of critical points.

Note that critical points of $I|_{S^{n-1}}$ occur in pairs since I is even. To prove Theorem 5.1 and treat other even functions, we need a tool by means of which we can classify and work with symmetric sets. With E a real Banach space, let $\mathcal{E}$ denote the family of subsets $A \subset E\setminus\{0\}$ such that A is closed in E and A is symmetric with respect to 0, i.e. $x \in A$ implies $-x \in A$. An index theory on $\mathcal{E}$ is a mapping $i : \mathcal{E} \to \mathbb{N} \cup \{\infty\}$ with the following properties:

$$\left.\begin{array}{l}
1^\circ \text{ (Normalization): For } x \in E\setminus\{0\},\ i(\{x,-x\}) = 1 \\
2^\circ \text{ (Mapping property): If } A,B \in \mathcal{E} \text{ and there is a map} \\
\quad \phi \in C(A,B) \text{ with } \phi \text{ odd, then } i(A) \le i(B) \\
3^\circ \text{ (Subadditivity): If } A,B \in \mathcal{E},\ i(A \cup B) \le i(A) + i(B) \\
4^\circ \text{ (Continuity property): If } A \in \mathcal{E} \text{ is compact,} \\
\quad i(A) < \infty \text{ and there exists a uniform neighborhood of} \\
\quad A,\ N_\delta(A) \equiv \{x \in E \mid \|x-A\| \le \delta\} \text{ such that } i(A) = i(N_\delta(A)).
\end{array}\right\} \qquad (5.2)$$

For future reference, note the following two easy consequences of 1°–4°:

5° (Monotonicity): If $A,B \in \mathcal{E}$ and $A \subset B$, $i(A) \le i(B)$

6° If $A,B \in \mathcal{E}$, $i(\overline{A\setminus B}) \ge i(A) - i(B)$

To prove 5°, just take $\phi = id$ in 2° and for 6° note that $A \subset B \cup \overline{A\setminus B}$ and invoke 5° and 3°.

Several examples of index theories can be found in the literature, mainly for $\mathbb{Z}_2$ and S^1 symmetries. See e.g. [26]-

[28]. The simplest one and easiest to work with is that provided by the notion of genus [6], [29]. For $A \in E$, $A \neq \phi$, we say A has <u>genus</u> n, denoted by $\gamma(A) = n$, if there is an odd map $\phi \in C(A, R^n\backslash\{0\})$ and n is the smallest integer with this property. If there is no finite such n we set $\gamma(A) = \infty$. By definition, $\gamma(\phi) = 0$.

Some simple examples are in order. Suppose $A = V \cup (-V) \in E$ where $V \cap (-V) = \phi$. Then $\gamma(A) = 1$ since we can set $\phi(x) = 1$ for $x \in V$ and $\phi(x) = -1$ for $x \in -V$ making ϕ odd and in $C(A, R\backslash\{0\})$. In particular any finite set of points A can be so decomposed so $\gamma(A) = 1$. Thus if $\gamma(A) \geq 2$, A contains infinitely many points. As another example, suppose $A \in E$ is a connected set. Then $\gamma(A) \geq 2$ for otherwise $\gamma(A) = 1$ implying there exists $\phi \in C(A, R\backslash\{0\})$ with ϕ odd. But $\phi(x) > 0$ for some $x \in A$ and $\phi(-x) < 0$. Since $\phi(A)$ is connected, this shows $0 \in \phi(A)$, a contradiction.

We will show that γ satisfies 1°-4° of (5.2). Indeed 1° is clear. To prove 2°, suppose $\gamma(B) \equiv n < \infty$ or the result is trivial. Hence there is an $f \in C(B, R^n\backslash\{0\})$ and odd. The function $f \circ \phi \in C(A, R^n\backslash\{0\})$ and is odd, so $\gamma(A) \leq n = \gamma(B)$. For 3°, let $\gamma(A) = m$, $\gamma(B) = n$, again both finite. Then there is a $\phi \in C(A, R^m\backslash\{0\})$, $\psi \in C(B, R^n\backslash\{0\})$, with ϕ and ψ odd. Using e.g. the Tietze Extension Theorem, continuously extend ϕ, ψ to $\hat{\phi} \in C(E, R^m)$, $\hat{\psi} \in C(E, R^n)$. By taking odd parts if necessary, we can assume $\hat{\phi}$, $\hat{\psi}$ are odd. Letting $f = (\hat{\phi}, \hat{\psi})$, $f \in C(A \cup B, R^{m+n}\backslash\{0\})$ and is odd. Hence $\gamma(A \cup B) \leq m + n = \gamma(A) + \gamma(B)$ and 3° holds. Lastly if A is compact, there are finitely many sets $A_j = B_{r_j}(x_j) \cup B_{r_j}(-x_j)$ with $r_j < \|x_j\|$ such that $A \subset \bigcup_1^k A_j$. By an above example, $\gamma(A_j) = 1$ so by 3°, $\gamma(A) < \infty$. If $\gamma(A) = m$, there is a $\phi \in C(A, R^m\backslash\{0\})$ with ϕ odd. As in the proof of 3°, ϕ has an odd extension $\hat{\phi} \in C(E, R^m)$. Since $\phi \neq 0$ on A, $\hat{\phi} \neq 0$ on $N_\delta(A)$ for some $\delta > 0$. Hence $\gamma(N_\delta(A)) \leq m$. But by 5° of (5.2), $\gamma(N_\delta(A)) \geq \gamma(A)$ and hence $\gamma(N_\delta(A)) = m = \gamma(A)$.

The index theory provided by γ is sufficient for our later purposes and therefore we will not discuss any others here. To prove Theorem 5.1 some additional preliminaries are needed.

Proposition 5.3: If $\mathcal{D}$ is a bounded, open neighborhood of 0 in $\mathbb{R}^k$ which is symmetric with respect to the origin and $\psi \in C(\partial\mathcal{D}, \mathbb{R}^j)$ with ψ odd and $j < k$, then there exists $\xi \in \partial\mathcal{D}$ such that $\psi(\xi) = 0$.

Proof: The proof can be found in [9].

Proposition 5.4: Let $\mathcal{D} \subset \mathbb{R}^k$ be a bounded, open, symmetric neighborhood of 0 and $A \in \mathcal{E}$ with A homeomorphic to $\partial\mathcal{D}$ by an odd map h. Then $\gamma(A) = k$.

Proof: Since $h \in C(A, \partial\mathcal{D})$ with h odd, $\gamma(A) \le k$. If $\gamma(A) = j < k$, there is a map $\phi \in C(A, \mathbb{R}^j\backslash\{0\})$ with ϕ odd. But then $\psi \equiv \phi \circ h^{-1} \in C(\partial\mathcal{D}, \mathbb{R}^j\backslash\{0\})$ with ψ odd, contrary to Proposition 5.3.

An immediate consequence of Proposition 5.4 is

Corollary 5.5: If $A \in \mathcal{E}$ is homeomorphic to S^{n-1} by an odd homeomorphism, $\gamma(A) = n$.

As a final preliminary we need a version of Theorem 1.1 for S^{m-1} - see Remark 1.4 (i) - which we shall simply state. Note that if $\tilde{I} \equiv I\big|_{S^{n-1}}$, $\tilde{I}'(x) = I'(x) - (I'(x)x)x$. For $c, s \in \mathbb{R}$, let $\tilde{K}_c = \{x \in S^{n-1} | \tilde{I}(x) = c$ and $\tilde{I}'(x) = 0\}$ and $\tilde{A}_s = \{x \in S^{n-1} | \tilde{I}(x) \le s\}$.

Proposition 5.6: If $I \in C^1(\mathbb{R}^n, \mathbb{R})$, for any $c \in \mathbb{R}$ and neighborhood $\mathcal{O}$ of $\tilde{K}_c$, there is an $\varepsilon > 0$ and $\eta \in C([0,1] \times S^{n-1}, S^{n-1})$ such that

1° $\eta(0,\cdot) = id$

2° $\eta(1, \tilde{A}_{c+\varepsilon}\backslash\mathcal{O}) \subset \tilde{A}_{c-\varepsilon}$

3° If $\tilde{K}_c = \phi$, $\eta(1, \tilde{A}_{c+\varepsilon}) \subset \tilde{A}_{c-\varepsilon}$

4° If I is even, $\eta(1,\cdot)$ is odd.

Now we can carry out the

Proof of Theorem 5.1: Let $\tilde{\gamma}_k = \{A \subset S^{n-1} | \gamma(A) \ge k\}$, $1 \le k \le n$. Corollary 5.5 shows $\tilde{\gamma}_k \ne \phi$, $1 \le k \le n$. Define

$$c_k \equiv \inf_{A \in \tilde{\gamma}_k} \max_{u \in A} \tilde{I}(u), \qquad 1 \le k \le n \tag{5.7}$$

Note that $c_1 < \dots < c_n$ since $\tilde{\gamma}_1 \supset \dots \supset \tilde{\gamma}_n$. We will prove that c_k is a critical value of $\tilde{I}$, $1 < k < n$. This in itself is not sufficient to prove Theorem 5.1 since c_k may be a "degenerate" critical value, i.e. $c_k = \dots = c_{k+j-1}$ for some $j > 1$ with possibly fewer than j distinct pairsof critical points corresponding to c (and hence fewer than n pairs for $\tilde{I}$). To surmount this potential difficulty, it suffices to establish the following "multiplicity lemma".

<u>Lemma 5.8:</u> Suppose $c_k = \dots = c_{k+j-1} \equiv c$. Then $\gamma(\tilde{K}_c) > j$.

Thus if $j > 1$, by earlier remarks, $\tilde{K}_c$ contains infinitely many distinct points. Thus Theorem 5.1 is an immediate consequence of Lemma 5.8.

<u>Proof of Lemma 5.8:</u> Suppose $\gamma(\tilde{K}_c) < j$. Then by Proposition 5.2, there is a $\delta > 0$ such that $\gamma(N_\delta(\tilde{K}_c)) < j$. We invoke Proposition 5.6 to get ε and η as in that result. Choose $A \in \tilde{\gamma}_{k+j-1}$ such that

$$\max_A \tilde{I} < c + \varepsilon \tag{5.9}$$

Let $B \equiv \overline{A \backslash N_\delta(K_c)}$. By 6° of Proposition 5.2, $\gamma(B) > k - 1$ so $B \in \tilde{\gamma}_k$ as is $\eta(1,B)$ via 2° of Proposition 5.2. But then by 2° of Proposition 5.6,

$$\max_{\eta(1,B)} \tilde{I} < c_k - \varepsilon \tag{5.10}$$

contrary to (5.7).

<u>Remark 5.11:</u> Since $\{x,-x\} \in \tilde{\gamma}_1$ for any $x \in S^{n-1}$, it follows that $c_1 = \min \tilde{I}$. Moreover $c_n = \max \tilde{I}$ since S^{n-1} is the only set in $\tilde{\gamma}_n$. In fact if $A \in \tilde{\gamma}_n$ and $A \neq S^{n-1}$, there is an $x \in S^{n-1} \backslash A$. We can then project A to $H \backslash \{0\}$ where H is the normal hyperplane to x through 0 and then project the resulting set radially into $H \cap S^{n-1}$. Since these projections are odd continuous maps, by 2° of Proposition 5.2 and Corollary 5.5, $\gamma(A) < \gamma(H \cap S^{n-1}) = n - 1$. Consequently no such A exists and $\tilde{\gamma}_n = \{S^{n-1}\}$.

<u>Remark 5.12:</u> Another way to characterize critical values of $\tilde{I}$ is as

$$b_j = \sup_{A \in \gamma_j} \min_{x \in A} \tilde{I}(x), \quad 1 < j < n . \tag{5.13}$$

It is easy to see that $b_1 = c_n$ and $c_1 = b_n$. However it is an open question as to whether $b_j = c_{n-j+1}$ for $j \neq 1,n$. If genus is replaced by the cohomological index of [26], it can be shown that the above intermediate critical values are equal.

There have been many infinite dimensional generalizations of Theorem 5.1, see e.g. [2], [10]. We will give one next.

<u>Theorem 5.14:</u> Let E be an infinite dimensional Hilbert space and $I \in C^1(E,\mathbb{R})$ be even. If $\tilde{I} \equiv I|_{\partial B_1}$ satisfies (PS) and is bounded from below, then $\tilde{I}$ possesses infinitely many distinct pairs of critical points.

<u>Proof:</u> The proof of Theorem 5.14 is formally the same as that of Theorem 5.1 using the infinite dimensional analogue of Proposition 5.6 and the fact that j in Lemma 5.8 must be finite via (PS). We omit the details.

The requirement that $\tilde{I}$ satisfies (PS) in Theorem 5.14 is too restrictive for applications. Consider e.g. the following problem

$$\begin{aligned} -\Delta u &= \lambda p(x,u), \quad x \in \Omega \\ u &= 0 \qquad\quad\;\;\, , \quad x \in \partial\Omega \end{aligned} \tag{5.15}$$

where p is odd, satisfies (p_1), (p_2),

(p_7) $p(x,\xi)$ is odd in ξ

and

(p_6') $\xi p(x,\xi) > 0$ if $\xi \neq 0$.

For $u \in E = W_0^{1,2}(\Omega)$, let

$$I(u) \equiv -\int_\Omega P(x,u)dx \tag{5.16}$$

Then $I \in C^1(E,\mathbb{R})$, is even, and is weakly continuous via Proposition 1.10. If $\tilde{I} \equiv I|_{\partial B_1}$, at a critical point u of $\tilde{I}$ we have

$$-\int_\Omega p(x,u)\phi dx + \left(\int_\Omega p(x,u)udx\right) \int_\Omega \nabla u \cdot \nabla\phi dx = 0$$

for all $\phi \in E$. Therefore u is a weak solution of (5.15) with $\lambda = \left(\int_\Omega p(x,u)udx\right)^{-1}$ provided that the λ term is finite and this will

be the case via (p_6') since $u \in \partial B_1$. The weak continuity of I implies $\tilde{I}$ is bounded from below (and above). Unfortunately $\tilde{I}$ does not satisfy (PS). In fact let (u_m) be any sequence in ∂B_1 such that $u_m \rightharpoonup 0$. Then $\tilde{I}(u_m)$ is certainly bounded. Moreover

$$\tilde{I}'(u_m) = I'(u_m) - (I'(u_m)u_m)u_m + I'(0) = 0$$

via (p_7) and Proposition 1.10. Since we can choose (u_m) so that $u_m \not\to 0$, (PS) is not satisfied.

All is not lost, however for recall Remark 1.4 (iii). One does not need (PS) globally but only a local version thereof. Our above remarks show the local version fails for this example when $c = 0$. Now suppose $c \neq 0$ and $(u_m) \subset \partial B_1$ is a sequence such that $\tilde{I}(u_m) \to c$ and $\tilde{I}'(u_m) \to 0$. Since (u_m) is bounded, $u_m \rightharpoonup u$. Therefore $I(u_m) \to I(u) = c$ and $I'(u_m)u_m \to I'(u)u$ via Proposition 1.10. Since $c \neq 0$, (p_6') implies $u \neq 0$. Consequently the sequence of numbers

$$I'(u_m)u_m = -\int_\Omega p(x,u_m)u_m dx$$

is bounded away from 0 and

$$u_m = (I'(u_m)u_m)^{-1}(I'(u_m) - \tilde{I}'(u_m))$$

converges strongly. Thus "$(PS)_{loc}$" is satisfied. Since we define

$$c_k = \inf_{A \in \tilde{\gamma}_h} \sup_{u \in A} \tilde{I}(u), \qquad k \in N$$

(5.16) and (p_6') imply $c_k < 0$ for each $k \in \mathbb{N}$. Consequently our above remarks yield

<u>Theorem 5.17:</u> If p satisfies (p_1), (p_2), (p_6'), and (p_7), (5.15) possesses a sequence of distinct pairs of weak solutions, $(\lambda_k, \pm u_k)$, on $R \times \partial B_1$, where $\lambda_k = I'(u_k)u_k$.

<u>Remark 5.18:</u> Of course the same argument works for ∂B_r for all $r > 0$. It is also a good exercise to show that $c_k \to 0$ as $k \to \infty$.

Theorems 5.1 and 5.14 yield multiple critical points of constrained symmetric functionals. The next result treats an unconstrained situation.

Theorem 5.19 (Clark [11]): Let $I \in C^1(E,R)$ be even and satisfy (PS). If $I(0) = 0$,

(I_7) I is bounded from below,

and

(I_8) There is a $K \in \mathcal{E}$ such that $\gamma(K) = k$ and $\sup_K I < 0$,

then I possesses at least k distinct pairs of critical points with corresponding negative critical values.

Proof: Set $\Gamma_j = \{A \subset \mathcal{E} | \gamma(A) \geq j\}$ and define

$$c_j = \inf_{A \in \Gamma_j} \sup_{u \in A} I(u) \qquad 1 \leq j \leq k \tag{5.20}$$

Since $\Gamma_{j+1} \subset \Gamma_j$, $c_1 \leq \dots \leq c_k$ and by (I_7), $c_1 > -\infty$. In fact Proposition 1.8 implies $c_1 = \inf_E I$. Hypothesis (I_8) shows $c_k < 0$. That the numbers c_j are critical values of I and there are at least k distinct pairs of corresponding critical points then follows essentially as in the proof of Theorem 5.1 with Theorem 1.1 replacing Proposition 5.6. One must also use the fact that $I(0) = 0$. Hence in the analogue of Lemma 5.8 for the current setting, since $c < 0$, $0 \notin K_c$ and $K_c \in \mathcal{E}$. The remaining details are the same as earlier so we omit them.

Remark 5.21. Actually Clark proved a more general result in [11].

We will give two applications of Theorem 5.19. Consider first

$$\begin{aligned} -\Delta u &= \lambda(u + p(u)), \quad x \in \Omega \\ u &= 0 \qquad\qquad\quad\;\, , \quad x \in \partial\Omega \end{aligned} \tag{5.21}$$

Let (λ_j) denote the eigenvalues of the corresponding linear problem (4.13) as earlier.

Theorem 5.23 [29], [30]: Suppose p satisfies (p_1'), (p_3), (p_7) and $\xi + p(\xi)$ satisfies (p_5). If $\lambda > \lambda_k$, then (5.22) possesses at least k distinct pairs of nontrivial solutions.

Proof: Define $\bar{p}(\xi) = \xi + p(\xi)$ for $\xi \in [-r,r]$ and $\bar{p}(\xi) = 0$ otherwise. Then $\bar{p}$ satisfies (p_1'), (p_5), (p_7), and (p_2) with $s = 0$. With $\bar{p}$ so defined, any solution of (2.37) will satisfy (5.22). Thus it suffices to find solutions of (2.37) or critical points of I_λ as defined in (2.38). By (p_7), I_λ is even in u and by the proof of Theorem 2.36, $I_\lambda \in C^1(E,\mathbb{R})$, satisfies (PS), (I_7), and $I_\lambda(0) = 0$. Thus once we verify (I_8), Theorem 5.23 follows from Theorem 5.19 and Proposition 1.13. Let

$$K \equiv \{ \sum_{j=1}^{k} \alpha_j v_j \mid \sum \alpha_j^2 = \beta^2 \} \qquad (5.24)$$

where v_j is as in (4.13) and β is free for now. It is clear that K is homeomorphic to S^{k-1} by an odd homeomorphism. Therefore $\gamma(K) = k$ by Corollary 5.5. For β sufficiently small, $\bar{P}(u) = \frac{1}{2} u^2 + P(u)$ for $u \in K$. Consequently,

$$I_\lambda(u) = \int_\Omega \left[\frac{1}{2} |\nabla u|^2 - \lambda\left(\frac{u^2}{2} + P(u)\right)\right]dx$$

$$= \sum_{j=1}^{k} \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_j}\right)\alpha_j^2 - o(\beta^2) < 0$$

for $u \in K$ via (p_3) and our choice of λ. Thus I_λ satisfies (I_8) and the proof is complete.

For our final application in this section, we prove a version of Theorem 1.36 for odd p.

Theorem 5.25 [18]: Suppose p satisfies (p_7) and the hypotheses of Theorem 2.36. Then for each $k \in \mathbb{N}$, there exists a $\bar{\lambda}_k$ such that for each $\lambda > \bar{\lambda}_k$, (2.35) has at least k distinct pairs of solutions.

Proof: As in the proof of Theorem 2.36 it suffices to prove the result for the modified equation (2.37) and therefore to find critical points of $I_\lambda(u)$ as defined in (2.38). Moreover $I_\lambda(u)$ is even in u and satisfies the hypotheses of Theorem 5.19 except possibly (I_8). Define K by (5.24) where β is so small that $u \in K$ implies $\bar{P}(u(x)) = P(u(x)) > 0$ if $u(x) \neq 0$. It follows that

$$\inf_{K} \int_{\Omega} P(u)dx \equiv \alpha_k > 0$$

Thus if $\lambda > \overline{\lambda_k} \equiv \frac{1}{2} \beta^2 \alpha_k^{-1}$,

$$\max_{K} I_\lambda(u) \le \frac{1}{2} \beta^2 - \lambda \alpha_k < 0$$

and (I_8) is satisfied. Theorem 5.19 and our above remarks then give us at least k distinct pairs of nontrivial solutions of (2.35) which correspond to negative critical values of I_λ.

Remark 5.26: By using some of the ideas in the proof of Theorem 6.1, it follows that I_λ also has at least k distinct pairs of critical points at which I_λ is positive. Thus for $\lambda > \lambda_k$, (2.35) has at least $2k$ distinct pairs of solutions. Theorem 2.36 with p odd is just a special case of this result with $k = 1$. See e.g. [18] for more details.

§6. A SYMMETRIC VERSION OF THE MOUNTAIN PASS THEOREM

In this section we will prove a $\mathbb{Z}_2$ version of the Mountain Pass Theorem.

Theorem 6.1 [18], [4]: Let E be an infinite dimensional Banach space and $I \in C^1(E,\mathbb{R})$ be even and satisfy $I(0) = 0$ and (PS). Suppose further $E = V \oplus X$, with V finite dimensional and I satisfies (I_5) and

(I_9) For all finite dimensional $\tilde{E} \subset E$, there is an $R = R(\tilde{E})$ such that $I(u) \le 0$ for $u \in \tilde{E} \setminus B_{R(\tilde{E})}$.

Then I possesses an unbounded sequence of critical values.

Remark 6.2: The simplest example of (I_5) is provided by (I_1) with the trivial splitting $V = \{0\}$ and $X = E$.

Proof of Theorem 6.1: We will define a family of sets Γ_j and obtain critical values of I by minimaxing I over these sets. Suppose V is ℓ dimensional with a basis $e_1,\ldots,e_\ell$. Choose $e_{m+1} \notin \text{span}\{e_1,\ldots,e_m\} \equiv E_m$ for $m \ge \ell$. Set $R_m \equiv R(E_m)$ and $D_m = B_{R_m} \cap E_m$. Define

$$G_m = \{h \in C(D_m,E) | h \text{ is odd and } h = id \text{ on } \partial B_{R_m} \cap E_m\} \tag{6.3}$$

Note that $G_m \neq \emptyset$; $id \in G_m$. Now set

$$\Gamma_j = \{\overline{h(D_m\backslash Y)} | m > j,\ h \in G_m,\ Y \in \mathcal{E},\ \gamma(Y) \leq m - j\} \tag{6.4}$$

Proposition 6.5: The sets Γ_j possess the following properties:

1° $\Gamma_j \neq \emptyset$

2° (Monotonicity) $\Gamma_{j+1} \subset \Gamma_j$

3° (Invariance) If $\phi \in C(E,E)$ is odd and $\phi = id$ on $\partial B_{R_m} \cap E_m$ for all $m \in \mathbb{N}$, then $\phi : \Gamma_j \to \Gamma_j$

4° (Excision) If $B \in \Gamma_j$, $Z \in \mathcal{E}$, and $i(Z) \leq s < j$, then $\overline{B\backslash Z} \in \Gamma_{j-s}$

Proof: Since $G_m \neq 0$, 1° is trivial. If $B = \overline{h(D_m\backslash Y)} \in \Gamma_{j+1}$, then $m > j + 1 > j$ and $i(Y) \leq m - (j + 1) < m - j$. Hence $B \in \Gamma_j$ and 2° is satisfied. If ϕ is as in 3° and $B = \overline{h(D_m\backslash Y)} \in \Gamma_j$, then $\phi \circ h \in G_m$. Therefore $\phi \circ \overline{h(D_m\backslash Y)} \in \Gamma_j$ and we have 3°. Lastly if $B = \overline{h(D_m\backslash Y)} \in \Gamma_j$ and $Z \in \mathcal{E}$ with $i(Z) \leq s < j$, then

$$\overline{B\backslash Z} = \overline{h(D_m\backslash(Y \cup h^{-1}(Z)))} \tag{6.6}$$

Assuming (6.6) for the moment, we have $h^{-1}(Z) \in \mathcal{E}$ since h is odd and therefore $Y \cup h^{-1}(Z) \in \mathcal{E}$. Moreover by 3° and 2° of (5.2),

$$i(Y \cup h^{-1}(Z)) \leq i(Y) + i(h^{-1}(Z)) \leq i(Y) + i(Z) \leq m - (j - s)$$

so $\overline{B\backslash Z} \in \Gamma_{j-s}$. To prove (6.6), observe that if $b \in \overline{B\backslash Z}$, then $b = h(x)$, $x \in \overline{D_m\backslash Y\backslash h^{-1}(Z)} \subset \overline{D_m\backslash Y \cup h^{-1}(Z)}$, i.e.

$$\overline{B\backslash Z} \subset \overline{h(D_m\backslash(Y \cup h^{-1}(Z)))} \tag{6.7}$$

On the other hand if $b \in h(D_m\backslash Y \cup h^{-1}(Z))$, then $b \in h(D_m\backslash Y)\backslash Z \subset B\backslash Z \subset \overline{B\backslash Z}$, i.e.

$$h(D_m\backslash(Y \cup h^{-1}(Z))) \subset \overline{B\backslash Z} \tag{6.8}$$

Comparing (6.7) and (6.8) yields (6.6) and the Proposition.

Next define

$$c_j = \inf_{B \in \Gamma_j} \max_{u \in B} I(u), \quad j \in \mathbb{N} \tag{6.9}$$

By 2° of Proposition 6.5,

$$c_j \le c_{j+1} \tag{6.10}$$

<u>Proposition 6.11:</u> For $j > \ell$, $c_j \ge \alpha$.

<u>Proof:</u> Let $B = h(\overline{D_m \setminus Y}) \in \Gamma_j$ where $m \ge j$, $\gamma(Y) \le m - j$. By the definition of R_m, $I(u) \le 0$ if $u \in E_m \setminus B_{R_m}$ while $I \ge \alpha$ on $\partial B_\rho \cap X$. Since $m > \ell$, $X \cap D_m \neq \emptyset$ and therefore $\rho < R_m$. Let $\hat{\mathcal{O}} = \{x \in D_m | h(x) \in B_\rho\}$ and let $\mathcal{O}$ denote the component of $\hat{\mathcal{O}}$ containing 0. Since h is odd and $h = id$ on $\partial B_{R_m} \cap E_m$, $\mathcal{O}$ is a symmetric bounded open neighborhod of 0 in E_m. With the obvious identification between E_m and $\mathbb{R}^m$, $\gamma(\partial\mathcal{O}) = m$ by Proposition 5.4. If $x \in \partial\mathcal{O}$, $h(x) \in \partial B_\rho$ so if $\Theta = \{x \in D_m | h(x) \in \partial B_\rho\}$, $\gamma(\Theta) \ge \gamma(\partial\mathcal{O}) = m$ by 5° of (5.2). Therefore $\gamma(\overline{\Theta \setminus Y}) \ge j > \ell$ and $\gamma(h(\overline{\Theta \setminus Y})) > \ell$ via 6° and 2° of (5.2). Consider $h(\overline{\Theta \setminus Y}) \cap X$. If this set were empty, the projector P of E onto V lies in $C(h(\overline{\Theta \setminus Y}), V \setminus \{0\})$ and is odd. But then $\gamma(h(\overline{\Theta \setminus Y}) \le \ell = \dim V$ contrary to our above calculation. Consequently $h(\overline{\Theta \setminus Y}) \cap X \neq \phi$ and by our definition of Θ, this intersection lies on ∂B_ρ. Thus for each $B \in \Gamma_j$, there is a $w \in B$ such that $h(w) \in \partial B_\rho \cap X$ and by (I_5),

$$\max_B I \ge \alpha$$

from which $c_j \ge \alpha$ follows.

Now we can prove that c_j is a critical value of I for $j > \ell$. This and more follows from

<u>Proposition 6.12:</u> If $j > \ell$ and $c_j = \ldots = c_{j+q-1} \equiv c$, then $\gamma(K_c) \ge q$.

<u>Proof:</u> Since $c \ge \alpha$ by Proposition 6.11 and $I(0) = 0$, $0 \notin K_c$ so $K_c \in \mathcal{E}$. Moreover K_c is compact by (PS). If $\gamma(K_c) < q$, by 4° of (5.2), there is a $\delta > 0$ such that $\gamma(N_\delta(K_c)) = \gamma(K_c) < q$. We invoke

the Deformation Theorem with $\mathcal{O} = N_\delta(K_c)$ and $\bar{\varepsilon} = \frac{\alpha}{2}$ obtaining $\varepsilon \in (0,\bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1, A_{c+\varepsilon} \setminus \mathcal{O}) \subset A_{c-\varepsilon} \tag{6.13}$$

Choose $B \in \Gamma_{j+q-1}$ such that

$$\max_{u \in B} I(u) < c + \varepsilon \tag{6.14}$$

Since $B \in \Gamma_{j+q-1}$, $\overline{B \setminus \mathcal{O}} \in \Gamma_j$ by 4° of Proposition 6.5. The mapping $\eta(1,\cdot)$ is odd so by our choice of $\bar{\varepsilon}$, $\eta(1,\cdot) \in G_m$ for all $m \in \mathbb{N}$. Hence $\eta(1,\overline{B \setminus \mathcal{O}}) \in \Gamma_j$ by 3° of Proposition 6.5. Thus by (6.13)-(6.14),

$$\max_{u \in \eta(1,\overline{B \setminus \mathcal{O}})} I(u) < c - \varepsilon \tag{6.15}$$

contrary to (6.9) and the Proposition is proved.

To show $c_j \to \infty$ as $j \to \infty$ thereby completing the proof of Theorem 6.1, an indirect argument related to that of Proposition 6.12 will be employed. By (6.10), (c_j) is a monotone nondecreasing sequence. If (c_j) were bounded, there is a $\bar{c}$ such that $c_j \to \bar{c}$ as $j \to \infty$. If $c_j = \bar{c}$ for all large j, $\gamma(K_{\bar{c}}) = \infty$ via Proposition 6.12. But $\gamma(K_{\bar{c}}) < \infty$ by (PS) and 4° of (5.2). Consequently $c_j < \bar{c}$ for all $j \in \mathbb{N}$. Set

$$\mathcal{K} = \{u \in E \mid c_{\ell+1} \leq I(u) \leq \bar{c},\ I'(u) = 0\}$$

By Proposition 6.11, $\mathcal{K} \in \mathcal{E}$ and by (PS) and 4° of (5.2) again, there is a $\delta > 0$ such that $\gamma(\mathcal{K}) = \gamma(N_\delta(\mathcal{K})) \equiv q < \infty$. We invoke the Deformation Theorem with $\bar{\varepsilon} = \bar{c} - c_{\ell+1}$ and $\mathcal{O} = N_\delta(\mathcal{K})$ to get ε and η satisfying (6.13) with c replaced by $\bar{c}$. Choose $j > \ell$ such that

$$c_j > \bar{c} - \varepsilon \tag{6.16}$$

and $B \in \Gamma_{j+q}$ such that

$$\max_{B} I < \bar{c} + \varepsilon\,. \tag{6.17}$$

Arguing as in Proposition 6.12, it follows from our choice of $\bar{\varepsilon}$ that $\eta(1,\cdot) \in G_m$ for all $m \in \mathbb{N}$ and that $\eta(1,\overline{B\setminus\mathcal{O}}) \in \Gamma_j$. But then (6.17), (6.13) and (6.15) (with $c = \bar{c}$) are contrary to (6.16) and (6.9). The proof of Theorem 6.1 is complete.

Remark 6.18: (i) Note that the proof of Proposition 6.12 works whenever the sets $\Gamma_j \neq \phi$. In particular the argument can be used to obtain the stronger form of Theorem 5.25 mentioned in Remark 5.26.

(ii) The condition that $I(0) = 0$ can be weakened to $I(0) < \alpha$ but if this is violated the result may no longer be true. A simple counterexample is $I(u) = 1 - \|u\|^2$.

We conclude this section with a PDE application of Theorem 6.1 to (1.9).

Theorem 6.19 [4]: Suppose p satisfies (p_1), (p_2), (p_4) and (p_7). Then (1.9) has an unbounded sequence of weak solutions.

Proof: As we know from the proof of Theorem 2.18, (p_1), (p_2), (p_4) imply I as defined in (1.11) belongs to $C^1(E,\mathbb{R})$ and satisfies (PS) and $I(0) = 0$. By (p_7), I is even. Moreover (p_4) implies (2.20) which in turn gives (I_9). If (I_5) holds, Theorem 6.1 implies I has an unbounded sequence of critical values (c_k). If u_k is a critical point corresponding to c_k,

$$c_k = I(u_k) = \frac{1}{2}\|u_k\|^2 - \int_\Omega P(x,u_k)dx \tag{6.20}$$

If (u_k) were bounded in E, (p_2), the Sobolev Embedding Theorem, and (6.20) show c_k would also be bounded. Thus (u_k) must be unbounded in E.

To verify (I_5), let (λ_k), (v_k) be as in (4.13). The eigenfunctions (v_k) form a basis for E. Let $V = \mathrm{span}\{v_1,\ldots,v_\ell\}$ and $X = V^\perp$ where ℓ is free for the moment. By (p_2)

$$I(u) \geq \frac{1}{2}\|u\|^2 - \int_\Omega (a_5|u|^{s+1} + a_6)dx \tag{6.21}$$

By the Gagliardo-Nirenberg inequality [31], for all $u \in E$,

$$\|u\|_{L^{s+1}} \leq b\|u\|^a\|u\|_{L^2}^{1-a} \tag{6.22}$$

where b is independent of u and

$$\frac{1}{s+1} = a\left(\frac{1}{2} - \frac{1}{n}\right) + (1-a)\,\frac{1}{2} \tag{6.23}$$

Moreover if $u \in X$, eigenfunction expansions show

$$\lambda_{\ell+1} \|u\|^2_{L^2} \le \|u\|^2 \tag{6.24}$$

Substituting (6.22)-(6.24) into (6.21) yields

$$I(u) \ge \rho^2\left(\frac{1}{2} - a_5 b^{s+1} \lambda_{\ell+1}^{\frac{-(1-a)(s+1)}{2}} \rho^{s-1}\right) - a_7 \tag{6.25}$$

for all $u \in \partial B_\rho \cap X$. Choose ρ so that the quantity in the parentheses in (6.25) equals $1/4$. Thus determines $\rho = \rho_\ell$ as a function of ℓ. Since $\lambda_\ell \to \infty$ as $\ell \to \infty$, choose ℓ so large that $\rho_\ell^2 > 8a_7$. Then

$$I(u) \ge \frac{1}{8}\,\rho_\ell^2 \equiv \alpha$$

on $\partial B_{\rho_\ell} \cap X$ and (I_5) holds. The proof is complete.

Remark 6.26: (i) Note that unlike Theorem 2.18, (p_7) allows us to eliminate all growth hypotheses on p near 0. (ii) A slight variation on (6.20) shows (u_k) is bounded in L^∞.

§7. PERTURBATIONS FROM SYMMETRY

In the previous two sections the existence of multiple critical points of even functionals has been studied. In this final section the effect of perturbing such a functional by an additional term that destroys the symmetry will be considered. We will work in the context of the example just treated in §6. Thus consider

$$\begin{aligned} -\Delta u &= p(x,u) + f(x), \quad && x \in \Omega \\ u &= 0 \quad , && x \in \partial\Omega \end{aligned} \tag{7.1}$$

and the corresponding functional

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega (P(x,u) + f(x)u)\,dx \tag{7.2}$$

Theorem 7.3 [32]-[35]: If $f \in L^2$ and p satisfies (p_1), (p_4), (p_7), and (p_2) with s such that

$$\frac{\mu}{\mu - 1} < \frac{(n + 2) - s(n - 2)}{n(s - 1)}, \tag{7.4}$$

then (7.1) possesses an unbounded sequence of weak solutions.

Results related to Theorem 7.3 have been obtained by Bahri and Berestycki [32], Struwe [33], Dong and Li [34], and the author [35]. The development we follow is that of [35]. Some of the more technical aspects of the proof will only be sketched referring to [35] for the details. Whether the restriction on s given by (7.4) is essential is not known. Bahri [36] has proved that if p is a pure power satisfying (p_2), the conclusions of Theorem 7.3 hold for almost all $f \in L^2$.

As in §6 to prove Theorem 7.3, it suffices to prove I has an unbounded sequence of critical values. The argument we employ requires an estimate of an appropriate type on the deviation of I from evenness. In particular we need

$$|I(u) - I(-u)| \le \beta_1\left(|I(u)|^{1/\mu} + 1\right) \tag{7.5}$$

for all $u \in E$. By (7.2) and (p_7)

$$I(u) - I(-u) = 2\int_\Omega f(x)u\,dx \tag{7.6}$$

We do not believe that (7.6) implies (7.5). However it is possible to replace I by a modified functional Φ such that Φ satisfies (7.5) and all large critical values of Φ are critical values of I. To motivate the modified problem, observe first that by (p_4), there are constants $a_3, a_4, a_5 > 0$ such that

$$\frac{1}{\mu}(\xi p(x,\xi) + a_3) \ge P(x,\xi) + a_4 \ge a_5|\xi|^\mu \tag{7.7}$$

for all $\xi \in \mathbb{R}$.

Lemma 7.8: If u is a critical point of I, there is a constant a_6 such that

$$\int_\Omega (P(x,u(x)) + a_4)dx \le a_6(I^2(u) + 1)^{1/2} \tag{7.9}$$

where a_6 depends only on $\|f\|_{L^2}$.

<u>Proof:</u> By (p_4), at a critical point of I,

$$I(u) = I(u) - \frac{1}{2} I'(u)u$$

$$\ge \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_\Omega (up(x,u) + a_3)dx - \frac{1}{2} \|f\|_{L^2}\|u\|_{L^2} - a_7$$

$$\ge \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_\Omega (up(x,u) + a_3)dx - \varepsilon\|u\|^\mu_{L^\mu} - K(\varepsilon)\|f\|^\sigma_{L^2} - a_7$$

via the Hölder and Young inequalities where $\sigma^{-1} + \mu^{-1} = 1$, $\varepsilon > 0$ is arbitrary, and $K(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Choosing $\varepsilon = \frac{1}{2}\left(\frac{\mu}{2} - 1\right)a_5$, (7.10) yields (7.9).

<u>Remark 7.11:</u> Note that Lemma 7.8 and (7.9) give us a bound for critical points u of I in terms of the corresponding critical value $I(u)$.

Now let $\chi \in C^1(\mathbb{R},\mathbb{R})$ such that $\chi(t) = 1$ for $t \le 1$, $\chi(t) = 0$ for $t \ge 2$ and $-2 < \chi' < 0$ for $t \in (1,2)$. For $u \in E$, define $\zeta(u) = 2a_6(I^2(u) + 1)^{1/2}$ and

$$\psi(u) \equiv \chi(\zeta(u)^{-1} \int_\Omega (P(x,u) + a_4)dx) . \tag{7.12}$$

Finally set

$$\Phi(u) \equiv \frac{1}{2} \|u\|^2 - \int_\Omega (P(x,u) + \psi(u)f(x)u)dx \tag{7.13}$$

If u is a critical point of I, by (7.9), $\psi(u) = 1$ and $\Phi(u) = I(u)$, $\Phi'(u) = I'(u) = 0$. Conversely we have

<u>Proposition 7.14:</u> Under the hypotheses of Theorem 7.3, $\Phi \in C^1(E,\mathbb{R})$ and there is a constant $M > 0$ such that

1° Φ satisfies (PS) on $W \equiv \{u \in E | \Phi(u) \ge M\}$

2° If u is a critical point of Φ in W, then $\Phi(u) = I(u)$ and $\Phi'(u) = I'(u) = 0$

3° Φ satisfies (7.5) for all $u \in E$.

Proof: We refer to [35] for the details of 1°-2°. To prove 3°, by (7.13),

$$\Phi(u) - \Phi(-u) = -(\psi(u) + \psi(-u)) \int_\Omega fu\,dx$$

If u is not in the support of ψ, $\psi(u) \int_\Omega fu\,dx = 0$ while if u is in the support of ψ,

$$\int_\Omega (P(x,u) + a_4)dx \le 2\zeta(u) \tag{7.15}$$

due to the definition of χ and form of ψ. Therefore

$$\psi(u) \int_\Omega fu\,dx \le (4a_6)^{1/\mu} a_8 a_5^{-1/\mu} \psi(u) \|f\|_{L^2} (I(u)^2 + 1)^{1/2\mu} \tag{7.16}$$

$$\le \psi(u) a_9 (|I(u)|^{1/\mu} + 1)$$

by (7.7) and (7.15). Since by (7.2) and (7.13),

$$|I(u)| \le |\Phi(u)| + 2|\int_\Omega fu\,dx| \, , \tag{7.17}$$

combining (7.16) and (7.17) yields

$$\psi(u) \int_\Omega fu\,dx \le \psi(u) a_{10} (|\Phi(u)|^{1/\mu} + |\int_\Omega fu\,dx|^{1/\mu} + 1) \, . \tag{7.18}$$

An application of Young's inequality and similar estimates for the $\psi(-u)$ term then give (7.5).

Now we turn to the question of finding critical points for Φ. Let (λ_m), (v_m) be as in (4.13), $E_m \equiv \text{span}\{v_1,\dots,v_m\}$, and $E_m^\perp$ the orthogonal complement of E_m. By (2.20), there is an R_m such that $\Phi(u) < 0$ for $u \in E_m \backslash B_{R_m}$. Set $D_m = B_{R_m} \cap E_m$ and

$$\tilde{\Gamma}_m = \{h \in C(D_m, E) \,|\, h = id \text{ on } \partial D_m\} \tag{7.19}$$

We define a sequence of minimax values of Φ:

$$b_m \equiv \inf_{h\in\tilde{\Gamma}_m} \max_{u\in D_m} \Phi(h(u)), \quad m \in \mathbb{N} \tag{7.20}$$

If $f \equiv 0$, the arguments of §6 show b_m is a critical value of I,

but with the f term present, these numbers are not in general critical values of I. We can however get a lower bound for b_m.

Proposition 7.21: There is a constant $\bar{a} > 0$ such that for large m,

$$b_m > \bar{a}\, m^{\beta} \tag{7.22}$$

where $\beta = (n(s-1))^{-1}(n + 2 - (n - 2)s)$

Proof: Using the fact that $\lambda_m > \text{const}.m^{2/n}$ [37] for large m, a slight modification of the argument given to verify (I_5) in Theorem 6.17 yields (7.22). We omit the details.

Before obtaining critical values of Φ, a second set of comparison values must be introduced. Let

$$U_m = \{u = tv_{m+1} + w \mid t \in [0,R_{m+1}],\ w \in B_{R_{m+1}} \cap E_m,\ \|u\| < R_{m+1}\},$$

i.e. U_m is the "upper" half of D_{m+1} and let

$$\Lambda_m = \{H \in C(U_m,E) \mid H|_{D_m} \in \Gamma_m \text{ and } H(u) = u$$

$$\text{for } \|u\| = R_{m+1} \text{ and for } u \in (B_{R_{m+1}} \setminus B_{R_m}) \cap E_m\}$$

Set

$$c_m \equiv \inf_{H \in \Lambda_m} \max_{u \in U_m} \Phi(H(u)), \quad m \in \mathbb{N} \tag{7.24}$$

With M as defined in Proposition 7.14, we have

Proposition 7.25: If $c_m > b_m > M$, $\delta \in (0, c_m - b_m)$, and

$$\Lambda_m(\delta) \equiv \{H \in \Lambda_m \mid \Phi(H)|_{D_m} < b_m + \delta\}$$

then

$$c_m(\delta) \equiv \inf_{H \in \Lambda_m(\delta)} \max_{u \in U_m} \Phi(H(u)) \tag{7.26}$$

is a critical value of Φ with $c_m(\delta) > c_m$.

Proof: By the definition of b_m, $\Lambda_m(\delta) \neq \phi$ and $c_m(\delta) > c_m > M$ since $\Lambda_m(\delta) \subset \Lambda_m$. If $c_m(\delta)$ is not a critical value of Φ, let

$\bar{\varepsilon} = \frac{1}{2}(c_m - b_m - \delta)$ and invoke the Deformation Theorem to get ε, η as usual. Choose $H \in \Lambda_m(\delta)$ such that

$$\max_{u \in U_m} \Phi(H(u)) \le c_m(\delta) + \varepsilon \tag{7.27}$$

Observe that $\eta(1,H) \in C(U_m, E)$ and by our choice of $\bar{\varepsilon}$, $\eta(1,H(u)) = u$ for $\|u\| = R_{m+1}$ and for $u \in (B_{R_{m+1}} \setminus B_{R_m}) \cap E_m$ since $\Phi \le 0$ on these sets. Moreover on D_m, $\Phi(H(u)) \le b_m + \delta \le c_m - \bar{\varepsilon} \le c_m(\delta) - \bar{\varepsilon}$, again via our choice of $\bar{\varepsilon}$. Hence $\eta(1,H) = H$ on D_m and therefore $\eta(1,H) \in \Lambda_m(\delta)$. But by (7.27) and the properties of $\eta(1,\cdot)$,

$$\max_{u \in U_m} \Phi(\eta(1,H(u))) \le c_m(\delta) - \varepsilon \tag{7.28}$$

contrary to (7.26).

<u>Remark 7.29:</u> Note that as $\delta \to 0$, $\Lambda_m(\delta)$ becomes smaller. An interesting open question is what happens to $c_m(\delta)$ as $\delta \to 0$.

To complete the proof of Theorem 7.3, with the aid of Proposition 7.21 and 7.26, it suffices to show that $c_m > b_m$ for arbitrarily large values of m. We will prove

<u>Proposition 7.30:</u> Suppose $b_m = c_m$ for all large m. Then there is a constant $\omega > 0$ such that

$$b_m \le \omega m^{\mu/\mu-1} \tag{7.31}$$

for all large m.

Comparing (7.31) to (7.22), we see these inequalities are incompatible with (7.4). Thus $c_m = b_m$ for all large m is impossible and our proof of Theorem 7.3 is complete.

<u>Proof of Proposition 7.30:</u> Choose $\varepsilon > 0$ and $H \in \Lambda_m$ such that

$$\max_{u \in U_m} \Phi(H(u)) \le b_m + \varepsilon . \tag{7.32}$$

Extend H to D_{m+1} as $\hat{H}(u) = H(u)$ for $u \in U_m$ and $\hat{H}(u) = -H(-u)$ for $u \in -U_m$. Since H is odd, $\hat{H} \in C(D_{m+1}, E)$ and belongs to $\tilde{\Gamma}_{m+1}$. Hence

$$b_{m+1} \le \max_{u \in D_{m+1}} \Phi(\hat{H}(u)) \qquad (7.33)$$

By (iii) of Proposition 7.14 and our definition of $\hat{H}$, for each $u \in U_m$,

$$\Phi(\hat{H}(-u)) \le \Phi(\hat{H}(u)) + \beta_1(|\Phi(\hat{H}(u))|^{1/\mu} + 1) \qquad (7.34)$$

Hence by (7.32) and (7.34),

$$\max_{u \in -U_m} \Phi(\hat{H}(u)) \le b_m + \varepsilon + \beta_1((b_m + \varepsilon)^{1/\mu} + 1) \ , \qquad (7.35)$$

and consequently by (7.33),

$$b_{m+1} \le b_m + \varepsilon + \beta_1((b_m + \varepsilon)^{1/\mu} + 1) \ . \qquad (7.36)$$

Since $\varepsilon > 0$ is arbitrary,

$$b_{m+1} \le b_m + \beta_1(b_m^{1/\mu} + 1) \qquad (7.37)$$

for all large m. Now by a straightforward induction argument, (7.37) implies (7.31) and the proof is complete.

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This paper contains a slightly expanded version of the series of lectures given by the author at DD3, the Third International Symposium on Differential Equations and Differential Geometry held in Changchun, China during parts of August and September of 1982. The lectures describe some of the results obtained using minimax methods in critical point theory during the past several years and give applications of the abstract results to differential equations. In particular existence theorems are obtained for many boundary value problems for semilinear elliptic partial differential equations.

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